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On the flow field of a rapidly oscillating airfoil in a supersonic flow

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This paper examines the features of the flow field off the surface of an oscillating flat-plate airfoil immersed in a two-dimensional supersonic flow Although the exact linearized solution for a supersonic unsteady airfoil has been known for a long time, its expression in the form of an integral is not convenient for a physical interpretation. In the present paper, the quintessential features of the flow field are extracted from the exact solution by obtaining an asymptotic expansion in descending powers of a frequency parameter through the repeated use of the stationary-phase and steepest descent methods. It is found that the flow field consists of two dominant and competing signals: one is the acoustic ray or that component arising from Lighthill's 'convecting slab 'and the other is the leading-edge disturbance propagating as a convecting wavelet. The flow field is found to be divided into several identifiable regions defined by the relative magnitude of the signals, and the asymptotic expansions appropriate for each flow region are derived along with their parametric restrictions. Such intimate knowledge of the flow field in unsteady, supersonic flow is of interest for interference aerodynamics and related acoustic problems.

1. Introduction

The piston theory for an oscillating airfoil in a supersonic flow, as it is now called, was first clearly enunciated by Lighthill (1953). On the basis of Hayes' (1947) hypersonic approximation for a steady flow, he pointed out that the pressure at a point on an airfoil surface oscillating in high Mach number flow is uniquely determined by the instantaneous vertical velocity of the airfoil at the same point. Lighthill's piston theory is not restricted to the linear situation; it can even handle the problems involving large disturbances. In the linear theory, however, the point relationship becomes further simplified to the extent that the perturbed pressure is directly proportional to the airfoil motion.

Both the utility and limitations of the linear piston theory were investigated extensively by Landahl, Ashley & Mollo-Christensen (1955). They showed that, for the piston theory to be valid, the requirement of high Mach number flow can be replaced by the condition that either a frequency parameter or the product of the Mach number and the frequency parameter should be sufficiently high. Under these circumstances, the linearized governing equation can be approximated by a simplified form which yields the result of the piston theory as a

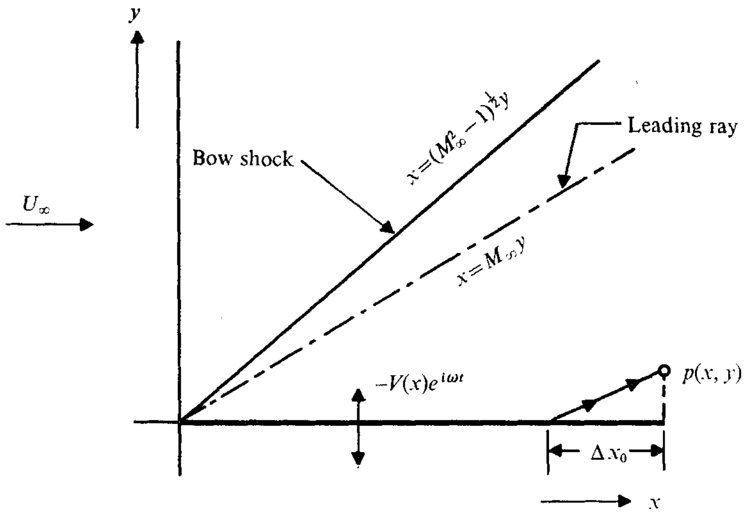

FIGURE 1. Definition sketch.

solution. By means of a numerical comparison of the piston theory with the exact values of aerodynamic coefficients, as tabulated by Garrick & Rubinow (1946), they found that even at a Mach number as low as $\frac{10}{7}$ the accuracy of the theory persists, provided that the frequency parameter is larger than 2. A historical survey of the piston theory and its applications can be found in Ashley & Zartarian (1956).

All these investigations dealt solely with the pressure on the airfoil surface. It seems equally interesting, however, to pose the following question. What is the relationship between the flow field off the airfoil surface and the motion of that surface? Knowledge of the flow field off the airfoil surface is not only of interest in its own right, but also has significance in problems involving aerodynamic interactions such as an airfoil in a wind tunnel, cascaded airfoils and wing–body interferences, and in related acoustic subjects. It is, in fact, a simple matter to construct a formal extension of the piston theory to the flow field off the airfoil from the following geometrical-acoustics viewpoint, which also agrees with Lighthill's original concept of a 'convecting slab'. If the frequency of the oscillating airfoil is sufficiently high, the disturbance created by its vertical motion is transmitted in the form of a ray, whose path is the vectorial sum of rectilinear propagation in a direction perpendicular to the airfoil and simultaneous convection in the downstream direction at the free-stream velocity.

Suppose that an airfoil oscillates about the x axis with a prescribed vertical velocity $V(x)\exp(i\omega t)$, where ω is the frequency (see figure 1). Let U_∞ and a_∞ be the free-stream and sound velocity far upstream. According to the foregoing argument, the signal received at a position $p(x, y)$ at time t should be the same as the signal originally emitted at some time, $t-\Delta t_0$, say, from the origin of the acoustic ray, located on the airfoil surface and at some distance Δx_0 upstream of x; i.e.

$$V(x-\Delta x_0)\exp[i\omega(t-\Delta t_0)].$$

During the time lag Δt_0, the signal has propagated vertically for a distance equal to y, with the sound velocity a_∞. Hence Δt_0 and y are related by

$$\Delta t_0 = y/a_\infty.$$

During the same interval, the signal has been convected in the downstream direction for a distance equal to Δx_0 at the free-stream velocity U_∞; that is to say

$$\Delta x_0 = U_\infty \Delta t_0 = M_\infty y.$$

Consequently, the signal at the point $p(x, y)$ at time t is given by

$$V(x - M_\infty y) \exp [i\omega(t - y/a_\infty)]. \tag{1.1}$$

Since the above argument is based on the notions of geometrical acoustics, it is also possible to derive (1.1) in a more formal manner as a high frequency limit. Hanin (1960), for example, applied the WKB method to the linearized governing equation and derived (1.1) as the leading term of the asymptotic series in powers of a frequency parameter. (It will be shown, however, that such a formal procedure appears to give inadequate results for the next higher order term on the surface and, in a flow field away from the airfoil, even for the leading term.)

According to (1.1), the geometrical acoustic rays, or the disturbances resulting from the extended form of the piston theory, propagate along the lines

$$x - M_\infty y = \text{constant}$$

and vanish outside the leading ray defined by $x = M_\infty y$. Since the Mach line emanating from the leading edge is given by $x = (M_\infty^2 - 1)^{\frac{1}{2}} y$, the leading ray is contained within the Mach line (figure 1). At first glance, the disappearance of (1.1) outside the leading ray seems to imply that, as far as the leading term is concerned, there may exist an additional zone of silence between the Mach line and the leading ray. The questions that arise immediately are the following. What happens to the effect of the disturbance created by the bow shock? Will the pressure rise across the bow shock always be a higher order term than the term given by (1.1)? If not, then in what regime of the flow field does the extended form of the piston theory cease to be the leading term and how should it be corrected? These are some of the issues that the present paper will resolve.

In what follows, we restrict the discussion to the two-dimensional problem and start from the exact linearized solution for the flow around an oscillating airfoil. To be sure, such a linearized treatment suffers from an inherent breakdown in a region sufficiently far from the airfoil. However, both in the near and the intermediate region, the linearized solution can still be expected to give a reasonably accurate description of the flow. By the repeated use of the stationary-phase and/or steepest-descent method, an asymptotic expansion of the linearized solution will be constructed as a descending series in a frequency parameter; the parameter to be used is a compressible reduced frequency based on the hyperbolic radius defined as $\omega\rho/a_\infty m^2$, where $\rho = (x^2 - m^2y^2)^{\frac{1}{2}}$ and $m = (M_\infty^2 - 1)^{\frac{1}{2}}$. Although the evaluation of an asymptotic series for a point on the airfoil surface is not of primary concern for the aforementioned purpose, because of its simple derivation it provides a useful check on the asymptotic series for the external flow field. Therefore, we shall first derive an asymptotic series for a field point *on* the airfoil surface, and then proceed to obtain an asymptotic expansion for a point *off* the surface, the latter being the main objective of the present paper; the expression so derived will be valid for large values of the frequency parameter, i.e. for high frequency oscillation and/or in the region sufficiently far away from the bow shock. We shall also examine how such flow fields are interconnected to the flow field in the vicinity of the bow shock.

2. Exact solution expressed as a double integral

The velocity potential for the flow field around an oscillating airfoil in a supersonic flow is given (e.g. Miles 1959, p. 50) by

$$\Phi(x,y) = \frac{1}{m}\operatorname{sgn}(y)\,H(x-my)\exp\left(-ik\frac{M_\infty}{m^2}x\right) \times \int_0^{x-my} V(\alpha)\exp\left(ik\frac{M_\infty}{m^2}\alpha\right) J_0\left[\frac{k}{m^2}\{(x-\alpha)^2-m^2y^2\}^{\frac{1}{2}}\right] d\alpha, \quad (2.1)$$

where

$$k = \omega/a_\infty, \quad m = (M_\infty^2-1)^{\frac{1}{2}}, \quad M_\infty > 1, \quad \operatorname{sgn}(y) = \pm 1 \quad \text{for} \quad y \gtrless 0. \quad (2.2)$$

J_0 is the zeroth-order Bessel function of the first kind and $H(x)$ is the Heaviside step function. To be more precise, $\Phi(x,y)$ is not the velocity potential itself but rather the amplitude of the perturbed velocity potential $\phi'(x,y,t)$; that is,

$$\phi' = \Phi(x,y)\exp(i\omega t).$$

The potential ϕ' satisfies the linearized governing equation

$$-m^2\frac{\partial^2\phi'}{\partial x^2}+\frac{\partial^2\phi'}{\partial y^2} = 2\frac{U_\infty}{a_\infty^2}\frac{\partial^2\phi'}{\partial x\,\partial t}+\frac{1}{a_\infty^2}\frac{\partial^2\phi'}{\partial t^2}, \quad (2.3)$$

with the following boundary condition:

$$\left.\frac{\partial\phi'}{\partial y}\right|_{y=0} = \frac{\partial\bar{y}}{\partial t}+U_\infty\frac{\partial\bar{y}}{\partial x} = -H(x)\,V(x)\exp(i\omega t), \quad (2.4)$$

where $\bar{y}(x,t)$ is the vertical co-ordinate prescribing the airfoil surface and $V(x)$ is the amplitude of the vertical velocity of the airfoil.

Since the frequency parameter k appears in both the arguments of the exponential and the Bessel function inside the integral sign of (2.1), we cannot directly apply such standard methods as the stationary-phase method to obtain an asymptotic form for large k. Likewise, direct substitution of the asymptotic form of J_0 inside the integral would not be valid, because its argument becomes zero at the upper end of the integral. The difficulty may be overcome by substituting for J_0 the expression

$$J_0(w) = \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\exp(iw\sin\theta)\,d\theta,$$

and changing the order of integration. $\Phi(x, y)$ is then recast as a double integral, and for $y > 0$ is given by

$$\Phi(x,y) = \frac{1}{\pi m} H(x-my) \exp\left(-ikx\frac{M_\infty}{m^2}\right)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} I\, d\theta, \tag{2.5}$$

where
$$I = \int_0^{x-my} V(\alpha) \exp\left[ik\frac{M_\infty}{m^2}\left\{\alpha + \frac{\sin\theta}{M_\infty}[(x-\alpha)^2 - m^2y^2]^{\frac{1}{2}}\right\}\right] d\alpha. \tag{2.6}$$

In the above integrand, k appears in the argument of the complex exponential and therefore the integral is now amenable to the direct application of the stationary-phase or saddle-point method. Although in the equation under consideration $V(x)$ is an arbitrary function of x ,it is assumed that $V(x)$ is continuous, and more important, that it does not influence the stationary-phase point or the saddle point determined by the exponential term in the integrand. This latter restriction precludes such $V(x)$ as $\exp(ikx)$, $\exp(ikx^2)$, etc., which, when combined with the exponential term in the integrand, would affect the position of the stationary point and complicate the subsequent analysis. However, the class of functions admissible as $V(x)$ does include, for example, the one corresponding to the following $\bar{y}(x,t)$, appearing in (2.4):

$$\bar{y}(x,t) = f(x)\exp(i\omega t), \tag{2.7}$$

where $f(x)$ is an arbitrary continuous function of x not involving ω or k, which is the case of interest for most purposes.

3. Asymptotic expansion for a point on the airfoil surface

To obtain the asymptotic expansion for a point on the airfoil surface, we set $y = 0$ in (2.6) and make the change of variables $\alpha = x\tau$. Then

$$I = x\int_0^1 V(x\tau)\exp\left[i\lambda\frac{M_\infty}{m^2}\left(\tau + \frac{\sin\theta}{M_\infty}(1-\tau)\right)\right] d\tau, \tag{3.1}$$

where
$$\lambda = kx = \omega x/a_\infty. \tag{3.2}$$

We seek the asymptotic behaviour for large values of $\lambda M_\infty/m^2$, and apply the method of stationary phase to (3.1). By substituting the resulting asymptotic form of I into (2.5) for $y = 0$ and evaluating some elementary integrals involved, we find

$$\begin{aligned}\Phi(x,0) = {} & (m\pi)^{-1}H(x)\exp\left(-i\lambda\frac{M_\infty}{m^2}\right)\left\{-i\pi m x V(x)\frac{1}{\lambda}\exp\left(i\lambda\frac{M_\infty}{m^2}\right)\right.\\ & + im^2xV(0)M_\infty^{-1}\frac{1}{\lambda}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left(1-\frac{\sin\theta}{M_\infty}\right)^{-1}\exp\left(i\lambda\frac{\sin\theta}{m^2}\right)d\theta\\ & + \pi m x^2 V'(x)M_\infty\frac{1}{\lambda^2}\exp\left(i\lambda\frac{M_\infty}{m^2}\right)\\ & - m^4x^2V'(0)M_\infty^{-2}\frac{1}{\lambda^2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left(1-\frac{\sin\theta}{M_\infty}\right)^{-2}\exp\left(i\lambda\frac{\sin\theta}{m^2}\right)d\theta\\ & \left. + O\left(\frac{1}{\lambda^3}\right)\right\} \quad \text{for} \quad \frac{\lambda M_\infty}{m^2} \gg 1.\end{aligned}$$

The stationary-phase method is again applied to the integrals above involving exponential functions for $\lambda/m^2 \gg 1$. A power series in $\lambda = kx$ follows at once:

$$\Phi(x,0) = \frac{H(x)}{k}\left\{-iV(x) + i(2\pi)^{-\frac{1}{2}}V(0)\left[(M_\infty+1)\exp\left(\frac{-ikx}{M_\infty+1} - i\frac{\pi}{4}\right)\right.\right.$$
$$\left.+(M_\infty-1)\exp\left(\frac{-ikx}{M_\infty-1}+i\frac{\pi}{4}\right)\right]\frac{1}{(kx)^{\frac{1}{2}}} + M_\infty V'(x)\frac{1}{k}$$
$$\left.+O(1/(kx)^{\frac{3}{2}})\right\} \quad \text{for} \quad kx/m^2 \gg 1. \tag{3.3}$$

The corresponding perturbed pressure p' is given by

$$\frac{p'}{\rho_\infty a_\infty^2} = -H(x)\exp(i\omega t)\left\{\frac{V(x)}{a_\infty} + (2\pi)^{-\frac{1}{2}}\frac{V(0)}{a_\infty}\frac{1}{(kx)^{\frac{1}{2}}}\right.$$
$$\left.\times\left[\exp\left(\frac{-ikx}{M_\infty-1}+i\frac{\pi}{4}\right) - \exp\left(\frac{-ikx}{M_\infty+1} - i\frac{\pi}{4}\right)\right] + O\left(\frac{1}{(kx)^{\frac{3}{2}}}\right)\right\} \quad \text{for} \quad \frac{kx}{m^2} \gg 1, \tag{3.4}$$

where ρ_∞ is the density far upstream. This is an expression valid for rapid oscillation and/or for a point far from the leading edge. The first term within the braces in this equation corresponds to the expression from the classical piston theory. The second term is proportional to the vertical velocity $V(0)$ at the leading edge of the airfoil and therefore represents the effect of the pressure rise across the bow shock formed at the leading edge. From the form of the arguments of the exponential terms, it is observed that such a disturbance propagates along the airfoil surface in the downstream direction at two different velocities equal, not surprisingly, to $U_\infty \pm a_\infty$, respectively. As the distance x from the leading edge increases, the leading-edge disturbance attenuates at a rate $x^{-\frac{1}{2}}$, showing a typical characteristic of a cylindrical diffraction wave. Also to be noted is that the term of order $(kx)^{-1}$ has vanished.

Hanin (1960) suggested the means to obtain the higher order correction terms to the piston theory, including the effect of airfoil surface curvature. He applied a WKB-type method to the original differential equation (2.3); in deriving an asymptotic expansion, he used formal standard expansions in integer powers of $(kx)^{-1}$. A comparison of his result, as applied to the present case of a flat plate, with (3.4) reveals that the first term is the same, as expected; the absence of the term of order $(kx)^{-1}$ is also common to both. In Hanin's result, however, there is no term of order $(kx)^{-\frac{1}{2}}$ corresponding to the second term of the present result or the cylindrical diffraction wave. This absence is a direct consequence of his formal expansion scheme in the integer powers of $(kx)^{-1}$.

On the other hand, when $V(x) =$ constant, the present result, including the term of order $(kx)^{-\frac{1}{2}}$, reduces to Candel's (1972) solution for the Sommerfeld diffraction problem in a supersonic flow. (Candel's solution and its connexion with the present result will be discussed in a wider context in § 6.) It appears, therefore, that results based on a formal expansion scheme in integer powers of $(kx)^{-1}$ such as that adopted by Hanin may not be adequate with respect to the higher order terms for a point on the airfoil surface.

So far we have examined the case of large values of a compressible reduced frequency. For small values of the reduced frequency, the pressure distribution can be obtained directly from (2.1) evaluated at $y = 0$. We thus find that

$$\frac{p'}{\rho_\infty a_\infty^2} = -H(x)\exp(i\omega t)\left\{\frac{V(x)}{a_\infty}\left(1-\frac{1}{M_\infty^2}\right)^{-\frac{1}{2}} + O(kx)\right\} \quad \text{for} \quad \frac{kxM_\infty}{m^2} \ll 1. \qquad (3.5)$$

This is an expression valid for slow oscillation and/or near the leading edge. The first term in braces in the above equation corresponds to Ackert's solution. Consequently, the pressure obtained by multiplying by a time factor $\exp(i\omega t)$ represents a quasi-steady solution, as expected. When the first term in braces in (3.5) is compared with the piston-theory term of (3.4), it is observed that, at large values of M_∞, the first term of (3.5) approaches the piston-theory term. For a highly supersonic flow, therefore, the pressure distribution at any point on the airfoil, including those points near the leading edge, can be approximated by the piston theory. On the other hand, in a low supersonic flow range, the pressure given by the piston theory loses its accuracy near the leading edge. This is precisely the reason why the aerodynamic coefficients computed from the piston theory alone become less accurate in a low supersonic range. At a low supersonic Mach number and in the case of flutter analysis pertinent to low frequencies, Morgan, Huckel & Runyan (1958) proposed the use of a quasi-steady solution at any point on the airfoil, its linear term being equal to the first term of (3.5).

4. Asymptotic expansion for a point off the airfoil surface

In (2.6) we introduce a new variable of integration σ defined by

$$\alpha = x - my\cosh\sigma.$$

It is also convenient to use the following hyperbolic co-ordinates (ρ, μ) instead of the Cartesian co-ordinates (x, y):

$$x = \rho\cosh\mu, \quad my = \rho\sinh\mu, \quad 0 \leqslant \rho, \mu, \qquad (4.1)$$

where ρ is the hyperbolic radius given by

$$\rho = (x^2 - m^2y^2)^{\frac{1}{2}}, \qquad (4.2)$$

and we note especially that $\rho = 0$ along the bow shock. With these substitutions, (2.6) becomes

$$I = \rho\sinh\mu\int_0^{\xi_2} V(\rho\cosh\mu - \rho\sinh\mu\cosh\sigma)\exp(\lambda h(\sigma))\sinh\sigma\, d\sigma, \qquad (4.3)$$

where
$$\lambda = k\rho, \qquad (4.4)$$

$$h(\sigma) = i\frac{M_\infty}{m^2}\left(\cosh\mu - \sinh\mu\cosh\sigma + \frac{\sin\theta}{M_\infty}\sinh\mu\sinh\sigma\right), \qquad (4.5)$$

and
$$\xi_2 = \cosh^{-1}(\coth\mu).$$

In § 3, the stationary-phase method was used at this point. However, in order to carry out the subsequent integration with respect to θ, it is more expedient to derive the asymptotic form of the above integral I by the steepest-descent method; the saddle point is given by

$$\sigma \equiv \xi_1 = \tanh^{-1}(\sin\theta/M_\infty). \tag{4.6}$$

Whether or not the saddle point is located within the path of integration of (4.3) is found to depend on the sign of $x - M_\infty y$ and on θ. At present it suffices to examine the case $x \geqslant M_\infty y$, or the flow field contained within the leading ray. For such a case, the position of the saddle point and the appropriate steepest-descent paths in the σ plane, where $\sigma = \xi + i\eta$, are shown schematically in figures 2(*a*) and (*b*) for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ and $-\frac{1}{2}\pi \leqslant \theta \leqslant 0$ respectively. Equation (4.3) is then written as

$$I = \begin{cases} I_{1+} + I_{2+} + I_{3+} + I_{4+} & \text{for} \quad 0 \leqslant \theta \leqslant \frac{1}{2}\pi, \quad (4.7a) \\ I_{1-} + I_{4-} & \text{for} \quad -\frac{1}{2}\pi \leqslant \theta \leqslant 0, \quad (4.7b) \end{cases}$$

where the integration paths are designated by the suffix. Substituting (4.7) into (2.5) and observing that (i) steepest paths $4+$ and $4-$ are the same and (ii) the integral of I_{1+} from 0 to $\frac{1}{2}\pi$ is equal to the integral of I_{1-} from $-\frac{1}{2}\pi$ to 0, we obtain

$$\Phi = \Phi_p + \Phi_{ld}, \tag{4.8}$$

where
$$\Phi_p \equiv 2\psi \int_0^{\frac{1}{2}\pi} I_{1+}\,d\theta + \psi \int_0^{\frac{1}{2}\pi} (I_{2+} + I_{3+})\,d\theta, \tag{4.9a}$$

and
$$\Phi_{ld} \equiv \psi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} I_{4+}\,d\theta, \tag{4.9b}$$

with
$$\psi = (\pi m)^{-1} H(x - my) \exp(-ikxM_\infty/m^2). \tag{4.9c}$$

As will be shown later, the term Φ_p turns out to be the piston-theory term and is therefore designated by a subscript p, while Φ_{ld} represents the leading-edge disturbance, hence the subscript ld. As stated, we first evaluate the integrand I appearing in (4.9*a*) and in (4.9*b*), which itself is an integral with respect to σ given by (4.3), through the method of steepest descent and then evaluate the integration with respect to θ by the method of stationary phase. In the course of this, we encounter a slight complication which elicits a comment. Both in I_{1+} and I_{4+}, the leading term of the asymptotic expressions of the first integral with respect to σ, which appears in the square-root, vanishes precisely at one of the end points of the subsequent integrations with respect to θ; in I_{1+}, this happens at any point in the flow field and in I_{4+}, this occurs only at those points located on the leading ray. With this in mind and by making use of the following definite integral,

$$\int_0^{\frac{1}{2}\pi} \pi^{\frac{1}{2}}\beta \sin\theta \exp[(\beta\sin\theta)^2]\,\mathrm{erfc}(\beta\sin\theta)\,d\theta = \tfrac{1}{2}\pi(1 - \exp\beta^2\,\mathrm{erfc}\,\beta),$$

which may be verified in a straightforward way by expanding the integrand as a power series in β and using the asymptotic form of the right-hand side, we obtain the following results for Φ_p and Φ_{ld}:

$$\Phi_p = -\rho i H(x - my)\, V(x - M_\infty y) \exp\left(-i\lambda \frac{\sinh\mu}{m}\right)\frac{1}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \quad \text{for} \quad \lambda \frac{M_\infty}{m^2} \gg 1 \tag{4.10}$$

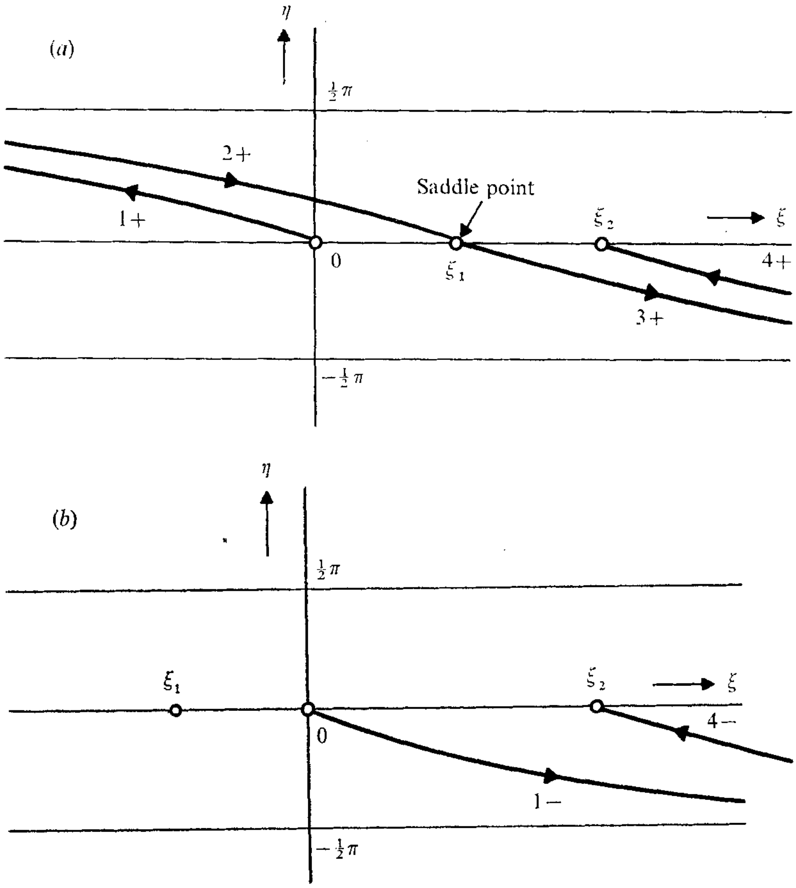

FIGURE 2. Steepest-descent paths in σ plane for (a) $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ and (b) $-\frac{1}{2}\pi \leqslant \theta \leqslant 0$. $\sigma = \xi + i\eta$. The steepest paths are as follows.

$$1\pm : \cos\eta\left(\cosh\xi - \frac{\sin\theta}{M_\infty}\sinh\xi\right) = 1, \quad \eta \gtrless 0.$$

$$2+,\ 3+ : \cos\eta\left(\cosh\xi - \frac{\sin\theta}{M_\infty}\sinh\xi\right) = \left[1-\left(\frac{\sin\theta}{M_\infty}\right)^2\right]^{-\frac{1}{2}}.$$

$$4\pm : \sinh\mu\cos\eta\left(\cosh\xi - \frac{\sin\theta}{M_\infty}\sinh\xi\right) = \cosh\mu - \frac{\sinh\theta}{M_\infty}, \quad \eta < 0.$$

and

$$\Phi_{ld} \sim -2^{-\frac{1}{2}} H(x-my)\rho V(0)\exp\left(-i\lambda\frac{M_\infty}{m^2}\cosh\mu\right) mM_\infty^{-\frac{1}{2}}$$
$$\times\left\{\exp\left(\frac{i\lambda}{m^2}-\frac{\pi i}{4}\right) b_+^{-\frac{1}{2}}\exp\left(\lambda\frac{M_\infty}{m^2}\Omega_+\right)\operatorname{erfc}\left[\left(\lambda\frac{M_\infty}{m^2}\Omega_+\right)^{\frac{1}{2}}\right]\right.$$
$$\left.+\exp\left(-\frac{i\lambda}{m^2}+\frac{\pi i}{4}\right) b_-^{-\frac{1}{2}}\exp\left(\lambda\frac{M_\infty}{m^2}\Omega_-\right)\operatorname{erfc}\left[\left(\lambda\frac{M_\infty}{m^2}\Omega_-\right)^{\frac{1}{2}}\right]\right\}\frac{1}{\lambda} \quad \text{for} \quad \frac{\lambda}{m^2} \gg 1, \tag{4.11}$$

where
$$b_{\pm} = 2i(\cosh\mu \mp 1/M_\infty), \tag{4.12a}$$
$$\Omega_{\pm} = \tfrac{1}{2}i(1 \mp \cosh\mu/M_\infty)^2(\cosh\mu \mp 1/M_\infty)^{-1}. \tag{4.12b}$$

Since (4.11) shows that Φ_{ld} is proportional to $V(0)$, Φ_{ld} represents in fact the disturbance at the leading edge. Note that in (4.11) there are two complementary error functions with different arguments, $\lambda M_\infty m^{-2}\Omega_{\pm}$. It can be shown that for $x \geqslant M_\infty y$
$$\min\{|\Omega_-|\} = \tfrac{1}{2}(1+1/M_\infty)^2(M_\infty + 1/M_\infty)^{-1},$$

and it follows that, at any point in the flow field within and on the leading ray, one argument of erfc, $|\lambda M_\infty m^{-2}\Omega_-|$, is always large for large $\lambda M_\infty/m^2$. Thus, erfc $[(\lambda M_\infty m^{-2}\Omega_-)^{\frac{1}{2}}]$ can always be expressed in its asymptotic form. On the other hand,
$$|\Omega_+| = \tfrac{1}{2}(1-\cosh\mu/M_\infty)^2(\cosh\mu - 1/M_\infty)^{-1}.$$

Noting that, along the leading ray ($x = M_\infty y$), (4.1) yields
$$\cosh\mu \equiv x(x^2 - m^2y^2)^{-\frac{1}{2}} = M_\infty; \tag{4.13}$$

we find that Ω_+ vanishes there. Thus, even for large $\lambda M_\infty/m^2$, the other argument of erfc, $|\lambda M_\infty m^{-2}\Omega_+|$, cannot always be large.

Consider, however, the flow field far from the leading ray where
$$|\lambda M_\infty m^{-2}\Omega_+| \gg 1;$$

both erfc $[(\lambda M_\infty m^{-2}\Omega_+)^{\frac{1}{2}}]$ and erfc $[(\lambda M_\infty m^{-2}\Omega_-)^{\frac{1}{2}}]$ can then be expressed in their asymptotic forms valid for large values of their arguments. We thus obtain
$$\begin{aligned}\Phi_{ld} &\sim i(2\pi)^{-\frac{1}{2}}H(x-my)\,m^2M_\infty^{-1}\rho\,V(0)\exp\left(-i\lambda\frac{M_\infty}{m^2}\cosh\mu\right)\\ &\times\left\{\exp\left(\frac{i\lambda}{m^2}-i\frac{\pi}{4}\right)\left(1-\frac{\cosh\mu}{M_\infty}\right)^{-1}+\exp\left(\frac{-i\lambda}{m^2}+i\frac{\pi}{4}\right)\right.\\ &\times\left.\left(1+\frac{\cosh\mu}{M_\infty}\right)^{-1}\right\}\frac{1}{\lambda^{\frac{3}{2}}}\quad\text{for}\quad\frac{\lambda}{m^2},\quad\left|\lambda\frac{M_\infty}{m^2}\Omega_+\right|\gg 1.\end{aligned} \tag{4.14}$$

On the other hand, in the neighbourhood of the leading ray, the leading term comes only from erfc $[(\lambda M_\infty m^{-2}\Omega_+)^{\frac{1}{2}}]$. From the asymptotic expansion of erfc (x) near $x = 0$, we have
$$\Phi_{ld} \sim 2^{-1}i\rho\,V(0)\exp(-iky)\frac{1}{\lambda}\quad\text{for}\quad\left|\lambda\frac{M_\infty}{m^2}\Omega_+\right| \ll 1,\quad\frac{\lambda}{m^2}\gg 1. \tag{4.15}$$

5. Description of the flow field

We now synthesize the solutions obtained so far and discuss the features of the flow field. From (4.8), the velocity potential is given by
$$\Phi = \Phi_p + \Phi_{ld}, \tag{5.1}$$

where the first term Φ_p is always given by (4.10), the leading term of which is of order λ^{-1}. Expressed in the Cartesian co-ordinate system, (4.10) becomes

$$\Phi_p = -iH(x-M_\infty y)\, V(x-M_\infty y) \exp(-iky)\frac{1}{k} + O\left(\frac{1}{(k\rho)^2}\right) \quad \text{for} \quad k\rho\frac{M_\infty}{m^2} \gg 1. \tag{5.2}$$

If the time factor $\exp(i\omega t)$ is restored, the leading term of the above equation becomes

$$-ik^{-1}\, V(x-M_\infty y)\exp[i\omega(t-y/a_\infty)],$$

which is in fact the extended form of the piston theory, derived in the introduction from physical arguments. We note again that the piston-theory term vanishes completely in the region outside the leading ray.

The second term, Φ_{ld} of (5.1), which represents the effect of the leading-edge disturbance, changes its magnitude considerably, depending on the field point. We divide the flow field contained within the leading ray into two regions, one far from the leading ray and the other near it, and begin with a discussion of the former region.

5.1. *Flow field inside and away from the leading ray*

In the region away from the leading ray ($x = M_\infty y$), Φ_{ld} is given by (4.14), and in such a region (5.1) becomes

$$\begin{aligned}\Phi = & -iH(x-M_\infty y)\, V(x-M_\infty y)\exp(-iky)/k \\ & + i(2\pi)^{-\frac{1}{2}} H(x-my)\, m^2\, V(0)\left[\exp\left(\frac{ik\rho}{m^2} - \frac{ikxM_\infty}{m^2} - i\frac{\pi}{4}\right)\right. \\ & \times \left(M_\infty - \frac{x}{\rho}\right)^{-1} + \exp\left(-\frac{ik\rho}{m^2} - \frac{ikxM_\infty}{m^2} + i\frac{\pi}{4}\right) \\ & \times \left.\left(M_\infty + \frac{x}{\rho}\right)^{-1}\right]\frac{1}{k(k\rho)^{\frac{1}{2}}} + O\left(\frac{1}{(k\rho)^2}\right) \quad \text{for} \quad \frac{k\rho}{m^2} \gg 1, \quad k\rho\frac{M_\infty}{m^2}|\Omega_+| \gg 1, \end{aligned} \tag{5.3}$$

where
$$\Omega_+ = \tfrac{1}{2}i(1-\cosh\mu/M_\infty)^2(\cosh\mu - 1/M_\infty)^{-1}. \tag{5.4}$$

The parametric restriction imposed on (5.3) implies that, in addition to the requirement that the point in the flow field has to be sufficiently far from the bow shock, the airfoil should be oscillating rapidly and/or the point be far from the leading ray. If y is set equal to zero in (5.3), it in fact reduces to (3.3), or the potential on the airfoil surface obtained in § 3. In that section, it was noted that the disturbance created at the leading edge propagates along the airfoil surface with two different velocities: $U_\infty \pm a_\infty$. It is only natural to expect that this observation can be generalized even to a point off the airfoil surface, which we next proceed to confirm.

With regard to the second term of (5.3), we focus our attention on the complex exponential terms with two different arguments: $\pm ik\rho/m^2 - ikxM_\infty/m^2$. These can be identified as corresponding to two diffraction wavelets emanating from the

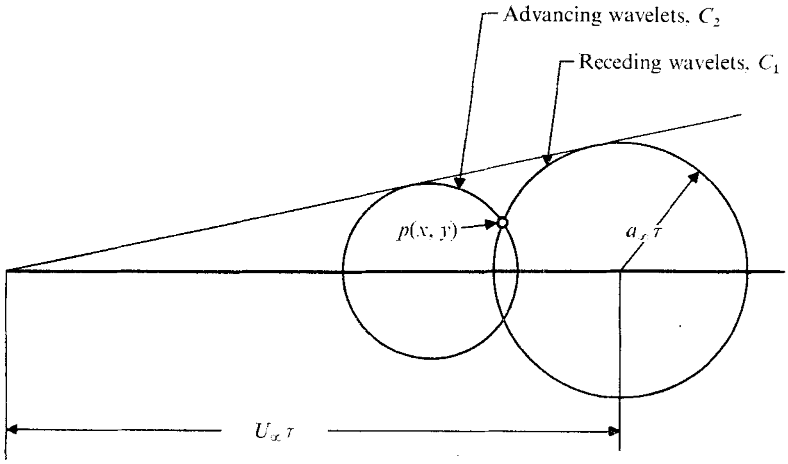

FIGURE 3. Two wavelets at $p(x,y)$.

leading edge and passing a given point $p(x,y)$ at time t. Two such familiar wavelets C_1 and C_2 are shown in figure 3. These diffraction wavelets represent the disturbance generated at the leading edge at a time $t-\tau$, before t; i.e.

$$\exp[i\omega(t-\tau)].$$

According to the elementary notion of wavelet propagation in a flowing medium, during the time interval τ, the centre of the wavelet has been convected a distance equal to $U_\infty\tau$, while its radius has increased to $a_\infty\tau$. The time delay τ corresponding to the point $p(x,y)$ is given by the relation $(x-U_\infty\tau)^2+y^2 = a_\infty^2\tau^2$, which yields

$$\tau = (\pm\rho + xM_\infty)/a_\infty m^2,$$

where the positive sign corresponds to the receding wavelet C_1 and the minus sign to the advancing wavelet C_2. Consequently the disturbance $\exp[i\omega(t-\tau)]$ is identified as

$$\exp(i\omega t)\exp[\pm ik\rho/m^2 - ikxM_\infty/m^2]. \tag{5.5}$$

These are indeed identical to the two complex exponential functions that appeared in the second term of (5.3). In connexion with the second term it is of interest to note that the decay rate of the leading-edge disturbance or a cylindrical diffraction wave, which, in a stationary medium, is inversely proportional to the square-root of the distance $r = (x^2+y^2)^{\frac{1}{2}}$ from the origin, can be simply obtained in the present case through the formal replacement of r by the hyperbolic radius ρ. Observe also that the amplitude $(M_\infty - x/\rho)^{-1}$ of the advancing wavelet is always larger than the amplitude $(M_\infty + x/\rho)^{-1}$ of the receding wavelet, as expected. Furthermore, since $M_\infty - x/\rho = 0$, formally at least, on the leading ray, the ratio of the amplitude of the advancing wavelet to that of the receding wavelet increases as the point $p(x,y)$ approaches the leading ray. This fact will be recalled in the next subsection.

5.2. *Flow field inside and near the leading ray*

In the neighbourhood of the leading ray, Φ_{ld} is given by (4.15), which has the same order, $O(\lambda^{-1})$, term as Φ_p. Thus Φ becomes

$$\Phi = -iH(x-M_\infty y)\, V(x-M_\infty y)\exp(-iky)/k+\tfrac{1}{2}iH(x-my)\, V(0)\exp(-iky)/k$$
$$+O\left(\frac{1}{(k\rho)^{\frac{3}{2}}}\right) \quad \text{for} \quad \frac{k\rho}{m^2} \gg 1, \quad k\rho\frac{M_\infty}{m^2}|\Omega_+| \ll 1. \quad (5.6)$$

We have pointed out that the first term represents the extended form of the piston theory. The second term can be identified again as the diffraction wavelet emanating from the leading edge. One can observe this by noting that one of the complex potentials of (5.3), that corresponding to the advancing wavelet, $\exp(ik\rho/m^2-ikxM_\infty/m^2)$, reduces, along the leading ray, to $\exp(-iky)$, which appears in the corresponding second term of (5.6). (The receding wavelet, the intensity of which has been found to become smaller than that of the advancing wavelet as the field point approaches the leading ray, would appear as the higher order term.) As another salient point, note that the same expression, $\exp(-iky)$, appears in the first term of (5.6) and furthermore, along the leading ray ($x = M_\infty y$), the amplitude $V(x-M_\infty y)$ of the first term is reduced to that of the leading-edge disturbance, $V(0)$. This implies that the geometrical-acoustics signal is also emitted from the leading edge and arrives at a point on the leading ray after the same time delay as that of the diffraction wavelet. In the limit when one approaches the leading ray from its inside, (5.6) becomes

$$\Phi = -\frac{i}{2}V(0)\exp(-iky)\frac{1}{k}+O\left[\frac{1}{(k\rho)^{\frac{3}{2}}}\right] \quad \text{for} \quad \frac{k\rho}{m^2} \gg 1, \quad \Omega_+ = 0. \quad (5.7)$$

Thus the leading ray is a demarcation line along which the leading term of the velocity potential is precisely equal to half of the value given by the piston theory and outside which the extended form of the piston theory vanishes identically.

In the results obtained by formal application of the WKB method, the leading term everywhere in the flow field is given by the piston-theory term corresponding to (5.2). It therefore fails to exhibit the foregoing intricate feature of the flow field off the airfoil surface; the inability of the formal WKB method to derive the higher order term on the airfoil surface was discussed in § 3.

If $k\rho M_\infty m^{-2}|\Omega_+|$ takes an intermediate value such that neither

$$k\rho M_\infty m^{-2}|\Omega_+| \gg 1 \quad \text{nor} \quad k\rho M_\infty m^{-2}|\Omega_+| \ll 1$$

is satisfied, the predominant term for $k\rho/m^2 \gg 1$ comes from the following combination of the piston-theory term and the advancing-wavelet term:

$$\Phi = -iH(x-M_\infty y)\, V(x-M_\infty y)\exp(-iky)/k$$
$$+2^{-\frac{1}{2}}iH(x-my)\,\rho\, V(0)\exp\left(\frac{ik\rho}{m^2}-\frac{ikxM_\infty}{m^2}\right)$$
$$\times\left[2\frac{M_\infty}{m^2}\left(\frac{x}{\rho}-\frac{1}{M_\infty}\right)\right]^{-\frac{1}{2}}\exp\left(k\rho\frac{M_\infty}{m^2}\Omega_+\right)\operatorname{erfc}\left[\left(k\rho\frac{M_\infty}{m^2}\Omega_+\right)^{\frac{1}{2}}\right]. \quad (5.8)$$

5.3. *Flow field outside the leading ray*

Outside the leading ray, the extended form of the piston-theory term vanishes abruptly. Since the velocity potential should be continuous across the leading ray, this implies that, in order to make up for such a loss, the intensity

$$+\tfrac{1}{2}iV(0)\exp(-iky)/k$$

(the second term of (5.6)) of the diffraction wavelet just inside the leading ray should suddenly increase to $-\frac{1}{2}iV(0)\exp(-iky)/k$ (the right-hand side of (5.7)) just outside it. This does not mean that the diffraction effect undergoes a discontinuous jump, for the higher order terms make the transition smooth.

Finally, consider the flow field in the neighbourhood of the bow shock ($x = my$). Such a flow is a particular case belonging to the more general class, subject to the parametric restrictions $kM_\infty m^{-2}(x-my) \ll 1$ and $k\rho/m^2 \ll 1$. Under such restrictions, (2.1) yields

$$\Phi(x,y) \sim \frac{1}{m}H(x-my)\exp\left(-ik\frac{M_\infty x}{m^2}\right) \times \int_0^{x-my} V(\alpha)\,d\alpha \quad \text{for} \quad \frac{kM_\infty}{m^2}(x-my) \ll 1, \quad \frac{k\rho}{m^2} \ll 1. \tag{5.9}$$

Setting y equal to zero reduces this to the quasi-steady solution on the airfoil surface discussed in § 3. In general, the equation under consideration shows that such a quasi-steady flow emanating from the airfoil surface propagates, without dispersion, along the Mach line. The parametric restrictions imposed on (5.9) imply that the airfoil should be oscillating slowly and/or the point in the flow field be sufficiently close to the bow shock. In the latter case, (5.9) becomes further simplified to the following form:

$$\Phi(x,y) \sim m^{-1}H(x-my)\exp(-ikM_\infty x/m^2)\,V(0)\,(x-my). \tag{5.10}$$

This again represents the leading-edge disturbance. The argument of the complex exponential appearing in the above equation is observed to be equal to that of the complex exponential in the second term of (5.3) if $\rho = (x^2-m^2y^2)^{\frac{1}{2}}$ is set equal to zero; this indicates that the quasi-steady disturbance created at the leading edge in fact propagates by means of the diffraction wavelet along the bow shock.

5.4. *Summary of flow field*

The entire structure of the flow may now be summarized as follows. The flow field is comprised of two dominant components, both of which are signals originating at the airfoil surface but propagating with distinctly different patterns. One of the signals corresponds to the geometrical-acoustics term or the extended form of the piston theory. It is emitted from an (any) oscillating point on the airfoil and propagates in the form of a ray; its path is rectilinear and the vectorial sum of the acoustic propagation in the direction normal to the airfoil surface and convection in the downstream direction at a free-stream velocity. It vanishes completely in the region between the leading ray and the initial Mach line originating from the leading edge. Inside the leading ray, this signal propagates without suffering any change in its initial intensity.

The other signal propagates in the form of the diffraction wavelets emanating from the leading edge†, whose growth is contained within the Mach cone. Its initial strength corresponds to the quasi-steady pressure rise across the bow shock formed at the leading edge; as the diffraction wavelet spreads out the effect of the leading-edge disturbance carried by the wavelet attenuates but, even on the same radius of a given wavelet, its intensity exhibits considerable variation along its circumference. (Such a variation should not be unexpected because of the strong dependence of the intensity on the angle of diffraction in the Sommerfeld diffraction problem with a stationary medium, certain connexions of which with the present problem will be discussed in the next section. However, in the present case the non-uniformity of the intensity is further enhanced by the fact that all the disturbances are contained in a narrower region downstream of the bow shock.) Consequently, the relative magnitude of these two competing signals varies in a rather complex way, dividing the flow field into several identifiable regions. In the following description of the flow field, we fix both the frequency ω and the Mach number and change only the location of the point in the flow field.

(i) The flow field away from the leading ray but contained within it is given by (5.3), provided that the compressible reduced frequency based on the hyperbolic radius is sufficiently large. Therein the geometrical-acoustic term dominates over the leading-edge disturbance term.

(ii) This situation includes the flow field on the airfoil surface as a particular case: the dominant signal can be identified to be the same as the one given by the classical piston theory.

(iii) As one approaches the leading ray, the intensity of the leading-edge disturbance increases, the leading term being given by (5.8). Along the leading ray, the leading-edge disturbance effect becomes of the same order as the piston-theory term, reducing the latter by half, as given by (5.7).

(iv) Beyond the leading ray, the signal due to the piston-theory term vanishes and the only contribution to the dominant term comes from the diffraction wavelet. As the wavelet spreads out, the quasi-steady disturbance created at the leading edge propagates without dispersion along the initial Mach line in the manner given by (5.10).

6. Relation of the present solution to the Sommerfeld diffraction problem

It is fitting at this point to note the connexion between the subject treated in the present paper and the Sommerfeld diffraction problem of a plane wave incident on a semi-infinite plate. In fact, certain features of the Sommerfeld diffraction problem, with a stationary medium, are found to occur also in the present solution. As an example, the phase lags of $\pm\frac{1}{4}\pi$ in the diffraction wavelet term of (5.3) are the same as the apparent phase jump discussed by Sommerfeld

† Any additional discontinuities in the surface velocity would similarly generate diffraction wavelets.

(1954, p. 262). Also the sudden increase of the diffraction wave intensity across the leading ray, pointed out in § 5.3, is similar to the diffraction band phenomena described by Sommerfeld and by Morse & Ingard (1968, p. 453).

Recently, Candel† (1972) obtained the solution of the Sommerfeld diffraction problem in a supersonic flow. He used the Fourier transform method. As the integration contour in the Fourier transform plane, an elliptical contour was chosen in the supersonic flow (hyperbolic field) rather than the hyperbolic contour suitable for the original Sommerfeld diffraction problem in a stationary medium (elliptic field). For normal incidence, the induced potential in his problem, or the total potential minus the incident wave, should agree with the present velocity potential. This situation of normal incidence is equivalent to setting $V(x) = \text{constant } (= ik)$ in (5.3) and (5.7) of the present analysis, which corresponds to the particular case of an airfoil executing flexural motion. The two results, those of the present paper and the calculation of Candel, are easily shown to be the same.

We reiterate at this point that our solution is based on an airfoil executing *arbitrary* motion. It is of special interest to observe that the generalization afforded by the present solution readily allows the formal replacement of $V(x) = \text{constant}$ by $V(x-M_\infty y)$ and $V(0)$ in the leading two terms of the asymptotic expansions.

7. Concluding remarks

It has been the intent of the present paper to display the features of the flow field off the surface of a rapidly oscillating airfoil immersed in a supersonic flow. Direct scrutiny of the exact integral representation involving a Bessel function is, however, not well suited for the examination of the underlying traits of the flow. The physical interpretation of the flow field, as summarily described in § 5.4, is found most effectively from the asymptotic expansions of the integral representation. A feature of the present analysis is that, by using a suitable integral formula for the Bessel function, the integral representation is transformed into a double integral to which one can repeatedly apply the method of steepest descent or stationary phase. In fact this enables one to derive the asymptotic expressions appropriate for the various regimes of the flow field in a natural and orderly manner and their parametric restrictions can be specified explicitly; the other methods, such as the WKB method, seem to miss some of the crucial features brought out by the present method. Another point worth re-emphasizing is that in the present analysis the airfoil motion is arbitrary; such generalized treatment offers the advantage that, not only can it obviously cope with a wide variety of airfoil motion, but also, while retaining the simple appearance of the asymptotic expansions, the physical meaning associated with each term of the series becomes clearly evident.

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